Optimal Weighted Poincaré and Log-Sobolev Inequalities for Cauchy Measures

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Abstract

In this paper, We establish the weighted Poincaré inequalities and Log-Sobolev inequalities for Cauchy distributions with optimal weight functions.

Keywords: Cauchy measure, weighted Poincaré inequality, weighted Log-Sobolev inequality.

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1 Introduction

A Borel probability μ on $\mathbb{R}^n$ is said to satisfy a weighted *Poincaré inequality* with weight function ω^2 (where ω is a fixed non-negative Borel measurable function), if there exists a constant $C > 0$ such that for every smooth function $f : \mathbb{R}^n \to R$ with gradient ∇f,

$$\mathrm{Var}_\mu(f) \le C \int |\nabla f|^2 \omega^2 d\mu,$$

where the variance of f w.r.t.μ is defined by

$$\mathrm{Var}_\mu(f) = \int \left(f - \int f d\mu \right)^2 d\mu.$$

Similarly, we say that a Borel probability μ on $\mathbb{R}^n$ satisfies a weighted *logarithmic Sobolev inequality* with weight function ω^2, if there exists a constant $C > 0$ such that for every smooth function $f : \mathbb{R}^n \to R$,

$$\mathrm{Ent}_\mu(f^2) \le C \int |\nabla f|^2 \omega^2 d\mu,$$

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where the entropy is defined by

$$\mathrm{Ent}_\mu(f^2) = \int f^2 \log f^2 d\mu - \left(\int f^2 d\mu\right) \log \left(\int f^2 d\mu\right).$$

The weighted functional inequalities are relatively weak inequalities. Although they can't deduce the exponential concentration of a probability measure for sure, they can also show some degree of decay for this measure, as can be seen in [5].

Consider the Cauchy measures:

$$d\mu_\beta(x) = \frac{(1+|x|^2)^{-\beta}}{c(n,\beta)} dx$$

where $\beta > \frac{n}{2}$, and $c(n,\beta)$ is the normalizing constant.

In [5], Bobkov and Ledoux proved that the Cauchy measures μ_β admit a weighted Poincaré inequality with the weight $1+|x|^2$ and the constant $\frac{(\sqrt{1+\frac{2}{\beta-2}}+\sqrt{\frac{2}{\beta-2}}\ )^2}{2(\beta-1)}$, as well as a weighted log-Sobolev inequalities with the weight $(1+|x|^2)^2$ and the constant $\frac{1}{\beta-1}$.

Comparing with the results in [5], the better weight functions are found by Hebisch and Zegarliński (in [13]). However, the authors just prove that the constants exist and are independent of β.

The aim of this paper is to give the optimal weight functions and the corresponding constants of the inequalities for the one-dimensional Cauchy measures in a different way. But it's a pity that we can't reach the similar results for the multi-dimensional case in the same way.

2 Main result

m1 **Theorem 2.1.** *(***one-dimensional weighted Poincaré inequality***) For any $\beta > 1/2$, the probability measure μ_β on $\mathbb{R}$ satisfy the following weighted Poincaré inequality: for any smooth function $f : \mathbb{R} \to \mathbb{R}$,*

$$\mathrm{Var}_{\mu_\beta}(f) \le C_\beta \int_{\mathbb{R}} |f^{'}(x)|^2 (1+x^2) d\mu_\beta(x),$$

where C_β has the same order with $\frac{1}{\beta}$. Moreover, the weight function is optimal in the sense of order.

m2 **Theorem 2.2.** *(***one-dimensional weighted log-Sobolev inequality***) For any $\beta > 1$, the probability measure μ_β on $\mathbb{R}$ satisfy the following weighted log-Sobolev inequality: for any smooth function $f : \mathbb{R} \to \mathbb{R}$,*

$$\mathrm{Ent}_{\mu_\beta}(f^2) \le C_\beta \int_{\mathbb{R}} |f^{'}(x)|^2 (1+x^2) \log(e+x^2) d\mu_\beta(x),$$

where C_β has the same order with $\frac{1}{\beta-1}$. Moreover, the weight function is optimal in the sense of order, that is, for any other function $\omega^2(x)$, if $\lim_{x\to+\infty} \frac{\omega^2(x)}{\log(e+x^2)} = 0$, then the Cauchy measure μ_β doesn't satisfy the weighted log-Sobolev inequality with the weight function $(1+x^2)\omega^2(x)$.

m3 **Theorem 2.3.** (**multi-dimensional weighted log-Sobolev inequality**) *For any $n \geq 6$, $\beta > n/2$, the probability measure μ_β on $\mathbb{R}^n$ satisfy the following weighted log-Sobolev inequality: for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$,*

$$\mathrm{Ent}_{\mu_\beta}(f^2) \leq C \int_{\mathbb{R}^n} |\nabla f(x)|^2 (1 + |x|^2) d\mu_\beta(x),$$

where $C = \dfrac{2}{n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n}}$ *is independent of β.*

Remark : The weight function is of course optimal. Because it has already been optimal for Poincaré type inequality.

Now we state the previous relative works by Bobkov and Ledoux ([5]), Hebisch and Zegarliński ([13]) :

thm2 **Theorem 2.4.** *([5]) For $\beta \geq n$, and any smooth bounded f on $\mathbb{R}^n$,*

$$\mathrm{Var}_{\mu_\beta}(f) \leq \frac{\left(\sqrt{1 + \frac{2}{\beta - 2}} + \sqrt{\frac{2}{\beta - 2}}\right)^2}{2(\beta - 1)} \int |\nabla f(x)|^2 (1 + |x|^2) \mu_\beta(x)$$

thm3 **Theorem 2.5.** *([5]) If $\beta \geq (n+1)/2$, $\beta > 1$, for any smooth bounded f on $\mathbb{R}^n$,*

$$\mathrm{Ent}_{\mu_\beta}(f^2) \leq \frac{1}{\beta - 1} \int |\nabla f(x)|^2 (1 + |x|^2)^2 \mu_\beta(x)$$

thm4 **Theorem 2.6.** *([13]) Assume μ_β is a probability measure on a n-dimensional manifold with metric d, and $d\mu_\beta = e^{-\beta \log(1+d) dx/Z}$ with $\beta \geq n$, $\beta > 1$. Suppose $Ric \geq 0$. Then for any $q \geq 1$, there are constants M_q, $c_q \in (0, +\infty)$, such that*

$$M_q \mu_\beta(|f - \mu_\beta(f)|^q) \leq \mu_\beta((1 + d)^q |\nabla f|^q)$$

and

$$\mu_\beta \left(|f|^q \log \frac{|f|^q}{\mu_\beta(|f|^q)} \right) \leq c_q \mu_\beta((1 + d)^q \log(e + d) |\nabla f|^q)$$

Theorem 2.1 does not give a better result than **Theorem 2.4** does. we just adapt another different way, and tell that the weight function is optimal.

From the comparison between **Theorem 2.2** with **Theorem 2.5**, it's clear that our result gives a better weight function. Moreover, the order of the constant isn't changed.

Contrast to **Theorem 2.6**, our results give the estimate of order for the constants.

In [13], their results still can derive a uniform weighted log-Sobolev inequality with weight $1 + |x|^2$ and constant c_q independent of β. In fact, we can also get the same result by Bakry-Émery criterion, i.e. getting $\Gamma_2 \geq \rho \Gamma$ for some $\rho > 0$.

3 Proofs of main results

3.1 One-dimensional weighted Poincaré inequality

thm01 **Theorem 3.1.** *([16]) Let μ, ν be Borel measures on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$ and $d\nu(x) = n(x)dx$. Let m be a median of μ. Let C_P be the optimal constant such that for every smooth function $f : \mathbb{R} \to \mathbb{R}$, one has*

$$\mathrm{Var}_\mu(f) \le C_P \int f'^2 d\nu.$$

Then $\max(b, B) \le C_P \le 4\max(b, B)$, *where*

$$b = \sup_{x<m} \mu((-\infty, x]) \int_x^m \frac{1}{n}, \qquad B = \sup_{x>m} \mu([x, \infty)) \int_m^x \frac{1}{n}.$$

Proof of Theorem 3.1: By the symmetry of the measure μ_β, the median m of μ_β is equal to 0. Define

$$b(\beta) := \sup_{\alpha\in(-\infty,0)} \left(\int_{-\infty}^x (1+y^2)^{-\beta} dy\right)\left(\int_x^0 (1+y^2)^{\beta-1} dy\right)$$

$$B(\beta) := \sup_{\alpha\in(0,+\infty)} \left(\int_x^{+\infty} (1+y^2)^{-\beta} dy\right)\left(\int_0^x (1+y^2)^{\beta-1} dy\right).$$

Clearly, by **Theorem 3.2** and symmetry, we just need to give an upper bound on $B(\beta)$. Since the point 0 doesn't make trouble in our calculation, we can reduce the estimate on $B(\beta)$ to that on $\widetilde{B}(\beta)$,

By the following simple estimate,

$$\int_x^{+\infty} (1+y^2)^{-\beta} dy = 2\int_{x^2}^{+\infty} (1+t)^{-\beta} t^{-1/2} dt \le \frac{2}{x}\frac{(1+x^2)^{-\beta+1}}{\beta-1},$$

and

$$\int_0^x (1+y^2)^{\beta-1} dy = 2\int_0^{x^2} (1+t)^{\beta-1} t^{-1/2} dy \le (1+x^2)^{\beta-1} x,$$

w get $B(\beta) \le \frac{2}{\beta-1}$.

However,

$$B(\beta) \ge \left(\int_{\frac{1}{\sqrt{\beta}}}^{\frac{2}{\sqrt{\beta}}} (1+y^2)^{-\beta} dy\right)\left(\int_{\frac{1}{2\sqrt{\beta}}}^{\frac{1}{\sqrt{\beta}}} (1+y^2)^{\beta-1} dy\right) \sim \frac{1}{\beta} \quad (\beta \to +\infty).$$

Thus we get the right order $\frac{1}{\beta}$ of C_β.

For any other even function $\omega^2(x)$ increasing in $x > 0$ (or increasing in $x \ge M$ for some

$M > 0$), and $\lim_{x\to+\infty}\frac{\omega^2(x)}{1+x^2}=0$,

$$
\begin{aligned}
B_{\omega^2}(\beta) :=& \sup_{\alpha\in(0,+\infty)}\left(\int_x^{+\infty}(1+y^2)^{-\beta}dy\right)\left(\int_0^x\frac{(1+y^2)^\beta}{\omega^2(y)}dy\right)\\
\geq& \sup_{x\in(M,+\infty)}\left(\int_x^{+\infty}(1+y)^{-2\beta}dy\right)\left(\int_M^x\frac{y^{2\beta}}{\omega^2(y)}dy\right)\\
\geq& \sup_{x\in(M,+\infty)}\frac{(1+x)^{1-2\beta}}{2\beta-1}\frac{1}{\omega^2(x)}\frac{x^{2\beta+1}-M^{2\beta+1}}{2\beta+1}\\
=&+\infty.
\end{aligned}
$$

Therefore, the weight function $1+x^2$ is optimal in the sense of order.

3.2 One-dimensional weighted Log-Sobolev inequality

Here we'll make use of the refined characterization from Barthe and Roberto [4]. Of course, we may also use that one from Bobkov, and Götze [6], after all, we can't give a sharp estimate on the logarithmic Sobolev constants. The refined characterization is stated as follows.

thm1 **Theorem 3.2.** *([4]) Let μ,ν be Borel measures on $\mathbb{R}$ with $\mu(\mathbb{R})=1$ and $d\nu(x)=n(x)dx$. Let m be a median of μ. Let C be the optimal constant such that for every smooth function $f:\mathbb{R}\to\mathbb{R}$, one has*

$$\mathrm{Ent}_\mu(f^2)\leq C\int f'^2d\nu.$$

Then $\max(b_-,b_+)\leq C\leq 4\max(B_-,B_+)$, *where*

$$
\begin{aligned}
b_+ &= \sup_{x>m}\mu([x,\infty))\log\left(1+\frac{1}{2\mu([x,\infty))}\right)\int_m^x\frac{1}{n},\\
B_+ &= \sup_{x>m}\mu([x,\infty))\log\left(1+\frac{e^2}{\mu([x,\infty))}\right)\int_m^x\frac{1}{n},\\
b_- &= \sup_{x<m}\mu((-\infty,x])\log\left(1+\frac{1}{2\mu((-\infty,x])}\right)\int_x^m\frac{1}{n},\\
B_- &= \sup_{x<m}\mu((-\infty,x])\log\left(1+\frac{e^2}{\mu((-\infty,x])}\right)\int_m^x\frac{1}{n}.
\end{aligned}
$$

Proof of Theorem 2.1: By the symmetry of the measure μ_β , the median m of μ_β is equal to 0. Moreover, in order to get our result, by **Theorem 3.2** we just need to give an upper bound on the following quantity,

$$
\begin{aligned}
S(\beta) :=& \sup_{x\in(0,+\infty)}\left(\int_x^{+\infty}(1+y^2)^{-\beta}dy\right)\log\left(1+\frac{c(1,\beta)}{\int_x^{+\infty}(1+y^2)^{-\beta}dy}\right)\\
&\cdot\left(\int_0^x\frac{(1+y^2)^\beta dy}{(1+y^2)\log(e+y^2)}\right).
\end{aligned}
$$

Clearly, the superior can't be obtained on the point 0. If the superior is taken on $x_\beta \in (0, +\infty)$. Let

$$I_1 = \int_{x\beta}^{+\infty} (1+y^2)^{-\beta} dy$$

$$I_2 = \int_0^{x_\beta} \frac{(1+y^2)^{\beta-1} dy}{\log(e+y^2)}$$

Then we'll discuss $S(\beta)$ under the following three sorts of situations by $\lim_{\beta\to+\infty}(1+(\frac{1}{\sqrt{\beta}})^2)^\beta = e$:
Convention: in the following we'll use the signal "$\sim$" to denote the same order as $\beta \to +\infty$.

Case 1: If $x_\beta = 0(\frac{1}{\sqrt{\beta}})$, i.e. $\lim_{\beta\to+\infty} \dfrac{x_\beta}{1/\sqrt{\beta}} = 0$:

$$I_1 \sim \frac{1}{\sqrt{\beta}}$$

$$I_2 \sim \int_0^{x_\beta} (1+y^2)^{\beta-1} dy \sim x_\beta$$

By the monotonicity of $x\log(1+\frac{C}{x})$ in $x > 0$, we have immediately

$$S(\beta) \sim \frac{x_\beta}{\sqrt{\beta}} = 0(\frac{1}{\beta}).$$

Case 2: If $x_\beta = O(\frac{1}{\sqrt{\beta}})$,

$$I_1 \sim \frac{1}{\sqrt{\beta}}$$

$$I_2 \sim \int_0^{x_\beta} (1+y^2)^{\beta-1} dy \sim \frac{1}{\sqrt{\beta}}$$

$$S(\beta) \sim \frac{1}{\beta}$$

Case 3: If $\frac{1}{\sqrt{\beta}} = 0(x_\beta)$,

$$I_1 \sim \frac{1}{\beta-1} \cdot \frac{1}{x_\beta(1+x_\beta^2)^{\beta-1}}$$

$$I_2 \sim \begin{cases} \dfrac{(1+x_\beta^2)^\beta}{\beta x_\beta} & \text{if } \{x_\beta\} \text{ is bounded} \\ \dfrac{1}{\log(e+x_\beta^2)} \dfrac{(1+x_\beta^2)^\beta}{\beta x_\beta} & \text{if } \{x_\beta\} \text{ is unbounded} \end{cases} \tag{3.1}$$

$$S(\beta) \sim \frac{1}{\beta}$$

Now there is the last case left that the superior is got as $x \to +\infty$. For that we might as well do it by reducing the estimate on $S(\beta)$ to that on $\widetilde{S}(\beta)$, defined by

$$\begin{aligned}\widetilde{S}(\beta) &:= \sup_{x\in(1,+\infty)} \left(\int_x^{+\infty} y^{-2\beta}dy\right) \log\left(1+\frac{1/(2\beta-1)}{\int_x^{+\infty} y^{-2\beta}dy}\right)\left(\int_1^x \frac{y^{2(\beta-1)}dy}{\log(e+y^2))}\right)\\ &= \sup_{x\in(1,+\infty)} \frac{x^{1-2\beta}}{2\beta-1}\log\left(1+x^{2\beta-1}\right)\int_1^x \frac{y^{2(\beta-1)}dy}{\log(e+y^2)}.\end{aligned}$$

By the basic formula of differential and integral, one can get readily

$$\begin{aligned}\frac{1}{2\beta-1}\frac{x^{2\beta-1}}{\log(e+x)} &\geq \frac{1}{2\beta-1}[\frac{y^{2\beta-1}}{\log(e+y)}]_1^x\\ &= \int_1^x \frac{y^{2(\beta-1)}}{\log(e+y)}\left(1-\frac{y}{(2\beta-1)(e+y)}\frac{1}{\log(e+y)}\right)dy\\ &\geq \int_1^x \frac{y^{2(\beta-1)}}{\log(e+y)}\left(1-\frac{1}{2\beta-1}\frac{1}{\log(e+y)}\right)dy\end{aligned}$$

When $2\beta-1>1$, i.e. $\beta>1$, we have

$$\begin{aligned}\int_1^x \frac{y^{2(\beta-1)}dy}{\log(e+y^2)} &\leq \int_1^x \frac{y^{2(\beta-1)}dy}{\log(e+y)}\\ &\leq \frac{1}{2\beta-2}\frac{x^{2\beta-1}}{\log(e+x)}\end{aligned}$$

Therefore,

$$\widetilde{S}(\beta) \leq \sup_{x\in(1,+\infty)} \frac{1}{(2\beta-1)(2\beta-2)}\frac{\log(1+x^{2\beta-1})}{\log(e+x)} \leq \frac{1}{2\beta-2}$$

From the discussion above, $\dfrac{1}{\beta}$ is the right order of log-Sobolev constants.

Moreover, for any other even function $\omega^2(x)$ increasing in $x>M>0$, and

$$\lim_{x\to+\infty}\frac{\omega^2(x)}{\log(e+x^2)}=0,$$

we have

$$\begin{aligned}\widetilde{S}_{\omega^2}(\beta) &:= \sup_{x\in(1,+\infty)} \left(\int_x^{+\infty} y^{-2\beta}dy\right) \log\left(1+\frac{1/(2\beta-1)}{\int_x^{+\infty} y^{-2\beta}dy}\right)\left(\int_1^x \frac{y^{2(\beta-1)}dy}{\omega^2(y)}\right)\\ &\geq \sup_{x\in(1,+\infty)} \frac{x^{1-2\beta}}{2\beta-1}\log\left(1+x^{2\beta-1}\right)\frac{1}{\omega^2(x)}\int_1^x y^{2(\beta-1)}dy\\ &= +\infty.\end{aligned}$$

As a result, the logarithmic-type weight function is optimal in the sense of order.

3.3 Multi-dimensional weighted log-Sobolev inequality

The Cauchy distribution can be represented in the following form:

$$d\mu_\beta(x) = \frac{e^{-\beta\log(1+|x|^2)}dx}{c(n,\beta)},\ x \in \mathbb{R}^n, \beta > n/2.$$

Let

$$V(x) := \beta\log(1+|x|^2),\ \omega^2(x) := 1+|x|^2,$$

we have

$$\nabla V = \frac{2\beta x}{1+|x|^2},\ \nabla^2 V = \frac{2\beta}{1+|x|^2} - \frac{4\beta x\otimes x}{(1+|x|^2)^2}$$

and

$$\nabla\omega^2 = 2x,\ \nabla^2\omega^2 = 2, \Delta\omega = 2n.$$

The generator, associated with the measure μ_β and the weight ω^2, is

$$\mathcal{L}_\omega := \omega^2\Delta + (\nabla\omega^2 - \omega^2\nabla V)\nabla. \tag{3.2}$$ diffusion

Note that μ_β is the invariant measure of the diffusion operator $\mathcal{L}_\omega$. We have the "Carré du champ" operator

$$\begin{aligned}\Gamma(f,g) &:= \frac{1}{2}(\mathcal{L}_\omega(fg) - f\mathcal{L}_\omega g - g\mathcal{L}_\omega f)\\ &= \omega^2\nabla f\nabla g.\end{aligned}$$

Define the Γ_2 curvature, see [2, 1, 14]

$$\Gamma_2(f,f) := \frac{1}{2}(\mathcal{L}_\omega\Gamma(f,f) - 2\Gamma(f,\mathcal{L}_\omega f)).$$

Proposition 3.3. *Assume that the dimension $n \geq 6$, we have the dimension curvature inequality holds, i.e. there exists positive constants ρ, such that*

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f), \tag{3.3}$$ CD

for all $f \in C_0^\infty(\mathbb{R}^n)$. ρ can be chosen to be $n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n}$ if necessary.

Proof. By the definition of Γ_2 curvature,

$$\begin{aligned}
\Gamma_2(f,f) &= \omega^4|\nabla\nabla f|^2 + \frac{\omega^2}{2}|\nabla f|^2\Delta\omega^2 + 2\omega^2\nabla f\nabla^2 f\nabla\omega^2 + \frac{1}{2}|\nabla f|^2|\nabla\omega^2|^2 \\
&\quad - \frac{1}{2}\omega^2|\nabla f|^2\nabla V\nabla\omega^2 - \omega^2\Delta f\nabla f\nabla\omega^2 - \omega^2\nabla f\nabla^2\omega^2\nabla f \\
&\quad + \omega^2(\nabla V\nabla f)(\nabla\omega^2\nabla f) + \omega^4\nabla f\nabla^2 V\nabla f \\
&= (1+|x|^2)^2|\nabla\nabla f|^2 + \big(2(\beta-1) + n(1+|x|^2)\big)|\nabla f|^2 \\
&\quad + 4(1+|x|^2)\nabla f\cdot\nabla\nabla f\cdot x - 2(1+|x|^2)\Delta f x\cdot\nabla f \\
&\overset{(i)}{\geq} (1+|x|^2)^2|\nabla\nabla f|^2 - 2\left(2+\frac{1}{\sqrt{n}}\right)(1+|x|^2)|\nabla\nabla f|(|x||\nabla f|) \\
&\quad + \big(2(\beta-1) + n(1+|x|^2)\big)|\nabla f|^2 \\
&= \left((1+|x|^2)|\nabla\nabla f| - \left(2+\frac{1}{\sqrt{n}}\right)|x||\nabla f|\right)^2 \\
&\quad + \left(\left(n-4-\frac{4}{\sqrt{n}}-\frac{1}{n}\right)|x|^2 + (2\beta-2+n)\right)|\nabla f|^2 \\
&\geq \min\left\{n-4-\frac{4}{\sqrt{n}}-\frac{1}{n}, 2\beta-2+n\right\}(1+|x|^2)|\nabla f|^2,
\end{aligned}$$

where $|\nabla\nabla f|$ is the Hilbert-Schmidt norm of $\nabla\nabla f$ and (i) follows from

$$(\Delta f)^2 \leq \frac{|\nabla\nabla f|^2}{n}.$$

If $n \geq 6$, there exists a positive constant $\rho := \rho(n) = n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n} > 0$ (independent on β), such that the dimension curvature inequality $CD(\rho,\infty)$, i.e. (3.3) holds. $\square$

By the above proposition, Theorem 2.3 is wellknown, see [2], also [17, 1, 14].

4 Appendix

a0 **Lemma 4.1.** *The normalized constant $c(1,\beta)$ has the same order with $\frac{1}{\sqrt{\beta}}$ as β goes to infinity.*

Proof.

$$\begin{aligned}
c(1,\beta) &= \int_{\mathbb{R}}(1+x^2)^{-\beta}dx \\
&= 2\int_0^{+\infty}(1+x^2)^{-\beta}dx \\
&= \int_0^{+\infty}(1+\tau)^{-\beta}\tau^{-1/2}d\tau \\
&= \int_0^{\frac{1}{\beta}}(1+\tau)^{-\beta}\tau^{-1/2}d\tau + \int_{\frac{1}{\beta}}^{+\infty}(1+\tau)^{-\beta}\tau^{-1/2}d\tau \sim \frac{1}{\sqrt{\beta}}.
\end{aligned}$$

This is because of the following facts:

$$\int_0^{\frac{1}{\beta}} (1+\tau)^{-\beta}\tau^{-1/2}d\tau \sim \int_0^{\frac{1}{\beta}} \tau^{-1/2}d\tau = \frac{2}{\sqrt{\beta}}.$$

and

$$\int_{\frac{1}{\beta}}^{+\infty} (1+\tau)^{-\beta}\tau^{-1/2}d\tau \leq \sqrt{\beta}\int_{\frac{1}{\beta}}^{+\infty} (1+\tau)^{-\beta}d\tau \sim \frac{1}{\sqrt{\beta}}.$$

Remark: Let $C(n,\beta) := \int_0^{+\infty}(1+r^2)^{-\beta}r^{n-1}dr$, then $c(n,\beta)$ has the same order with $C(n,\beta)$ as $\beta \to +\infty$. Next we give an estimate for $C(n,\beta)$.

$$C(n,\beta) = \frac{1}{2}\int_0^{+\infty} (1+\tau)^{-\beta}\tau^{\frac{n-2}{2}}d\tau$$

By the similar method, we can get

$$C(n,\beta) \sim \frac{1}{\beta^{\frac{n}{2}}}.$$

That is $c(n,\beta)$ has the order of $\dfrac{1}{\beta^{\frac{n}{2}}}$ as β large enough. □

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